JOURNAL OF COMPUTATIONAL PHYSICS 47, 321–329 (1982)

Note

Chebyshev Series Solution of the Controlled Duffing Oscillator

1. INTRODUCTION

In the last decennium,, various new computational techniques involving mathematical programming methods and the use of modern digital computers have been established especially for applications in optimal systems control [1–3]. Alternative techniques for the dynamic programming method [4] and Pontryagin's maximum principle method [5], the two most adequate techniques for solving optimal control problems, have been proposed. Recently, special attention has been devoted to the study of the controlled Duffing oscillator which is known to describe many important oscillating phenomena in nonlinear engineering systems [6, 7].

The aim of this paper is to introduce a direct computational technique to solve the controlled Duffing oscillator by taking into account the important advantages of the use of the Chebyshev polynomials in numerical analysis with regard to minimax principles and least squares techniques [8]. The method is based on the series expansion of the state function and the control strategy in Chebyshev polynomials having undetermined coefficients. One of the major advantages which the use of the Chebyshev polynomials provides is that the differential and integral expressions, which occur in the algorithm when approximating the given dynamical system, the boundary conditions, and the performance index, are readily transformed into simple algebraic expressions in the unknown coefficients.

The method of constrained extremum is applied, which consists of introducing some unknown Lagrange multipliers in conjunction with the constraint equations which are derived from the approached dynamical system and the boundary conditions. The controlled linear oscillator is investigated first in Sections 2 and 3. The Chebyshev approximations of high order are obtained by solving a linear system and are found to agree quite well with the exact solution which is available from Pontryagin's maximum principle method. Section 4 is devoted to the study of the controlled Duffing oscillator in Chebyshev series for which the determining equations for the unknowns are nonlinear algebraic equations. These equations are then solved by the generalized Newton–Raphson iterative method using some of the previous results for the controlled linear oscillator as starting values needed to initiate the iterative procedure.

0021-9991/82/080321–09$02.00/0

The results presented here are an extension to optimal control problems of the work of Urabe [9] and Van Dooren [10–14], among others, involving the use of orthogonal polynomials in the numerical computation of solutions to multipoint boundary value problems for ordinary differential equations.

2. The Controlled Linear Oscillator

Consider the optimum control of a linear oscillator governed by the differential equation

$$\ddot{x} + \omega^2 x = u, \tag{2.1}$$

in which a dot $(\cdot)$ means differentiation with respect to τ, where $-T \leqslant \tau \leqslant 0$ and T is specified. Equation (2.1) is equivalent to the dynamic state equations

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\omega^2 x_1 + u. \tag{2.2}$$

One wishes to control the state of this plant between the specified values $x_1(-T) = x_{10}$, $x_2(-T) = x_{20}$, and $x_1(0) = 0$, $x_2(0) = 0$, steering the state to the origin of the phase plane, such that the performance index

$$I = \tfrac{1}{2} \int_{-T}^{0} u^2 d\tau, \tag{2.3}$$

is minimized over all admissible control functions $u(\tau)$.

Pontryagin's maximum principle method [5] applied to this optimal control problem yields the following exact analytical solution representation

$$\begin{aligned}
x_1(\tau) &= (1/2\omega^2)[A\omega\tau \sin \omega\tau + B(\sin \omega\tau - \omega\tau \cos \omega\tau)], \\
x_2(\tau) &= (1/2\omega)[A(\sin \omega\tau + \omega\tau \cos \omega\tau) + B\omega\tau \sin \omega\tau], \\
u(\tau) &= A \cos \omega\tau + B \sin \omega\tau, \\
I &= (1/8\omega)[2\omega T(A^2 + B^2) + (A^2 - B^2) \sin 2\omega T - 4AB \sin^2 \omega T],
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
A &= 2\omega[x_{10}\omega^2 T \sin \omega T - x_{20}(\omega T \cos \omega T - \sin \omega T)]/(\omega^2 T^2 - \sin^2 \omega T), \\
B &= 2\omega^2[x_{20} T \sin \omega T + x_{10}(\sin \omega T + \omega T \cos \omega T)]/(\omega^2 T^2 - \sin^2 \omega T).
\end{aligned} \tag{2.5}$$

3. The Chebyshev Approach and its Results

In view of the use of the Chebyshev polynomials $T_n(t)$, which are defined on the interval $[-1, 1]$, the transformation $\tau = \frac{1}{2}T(t-1)$ is introduced. The optimal control problem may then be restated as follows: Minimize

$$I = \tfrac{1}{4}T \int_{-1}^{1} u^2\, dt, \tag{3.1}$$

subject to

$$\ddot{x} = \tfrac{1}{4}T^2(-\omega^2 x + u), \qquad (\dot{} = d/dt), \tag{3.2}$$

with

$$x(-1) = x_{-1}, \qquad \dot{x}(-1) = \dot{x}_{-1}, \qquad x(1) = 0, \qquad \dot{x}(1) = 0. \tag{3.3}$$

Let us determine an approximate solution of this optimal control problem represented by a Chebyshev series of order m for both the state and the control:

$$\begin{aligned} x_m(t) &= \tfrac{1}{2}a_0 T_0(t) + \sum_{n=1}^{m} a_n T_n(t), \\ u_m(t) &= \tfrac{1}{2}b_0 T_0(t) + \sum_{n=1}^{m} b_n T_n(t), \end{aligned} \tag{3.4}$$

thus assuming that approximation by polynomials is appropriate. Otherwise, as discussed by Dahlquist and Bjorck [15] or Fox and Parker [8], preliminary transformations of variables or approximation by rational functions must be used.

The unknown coefficients $\alpha \equiv (a_0, a_1, \ldots, a_m)$ and $\beta \equiv (b_0, b_1, \ldots, b_m)$ in Eq. (3.4) are determined as follows: First one replaces the original system dynamics (3.2) by its approximate version

$$\ddot{x}_m(t) = \tfrac{1}{4}T^2[-\omega^2 x_m(t) + u_m(t)], \tag{3.5}$$

upon which a Chebyshev balance principle is applied, this consisting of equating the coefficients of the Chebychev polynomials $T_n(t)$ in this equation. Using a well-known property for the Chebyshev series [8], one has the following relations between the Chebyshev coefficients q_n of a continuously differentiable function $q(t)$ and the Chebyshev coefficients q_n'' of its second derivative $\ddot{q}(t)$

$$(n+1)\, q''_{n-2} - 2nq''_n + (n-1)\, q''_{n+2} - 4n(n^2-1)\, q_n = 0 \qquad (n = 2, 3, \ldots). \tag{3.6}$$

According to this property, the application of the Chebyshev balance principle to the approximated system dynamics (3.5) yields the $m+1$ equations

$$\begin{aligned} (n+1)\, A_{n-2} - 2nA_n + (n-1)\, A_{n+2} - 4n(n^2-1)\, a_n = 0 \quad & (n = 2, 3, \ldots, m), \\ (n+1)\, A_{n-2} - 2nA_n + (n-1)\, A_{n+2} = 0 \quad & (n = m+1, m+2), \end{aligned} \tag{3.7}$$

with $A_{m+1} = A_{m+2} = A_{m+3} = A_{m+4} = 0$. The coefficients A_n are the Chebyshev coefficients of the right-hand side of Eq. (3.5).

By the use of the properties $T_n(-1) = (-1)^n$ and $T_n(1) = 1$, the boundary conditions (3.3) are approached by

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{m} (-1)^n a_n - x_{-1} = 0, \qquad \sum_{n=1}^{m} (-1)^{n+1} n^2 a_n - \dot{x}_{-1} = 0,$$
$$\tfrac{1}{2}a_0 + \sum_{n=1}^{m} a_n = 0, \qquad \sum_{n=1}^{m} n^2 a_n = 0. \tag{3.8}$$

For the approximation of the performance index characterized by a general function $g(x, u, t)$, we consider the expression

$$J(\alpha, \beta) = \int_{-1}^{1} g[x_m(t), u_m(t), t]\, dt. \tag{3.9}$$

Let

$$B_n(\alpha, \beta) = \frac{2}{\pi} \int_{-1}^{1} (1 - t^2)^{-1/2} g[x_m(t), u_m(t), t]\, T_n(t)\, dt \tag{3.10}$$

represent the Chebyshev coefficients of $g[x_m(t), u_m(t), t]$, then according to a well-known theorem for the integration of a Chebyshev series [8], one has

$$J(\alpha, \beta) = B_0(\alpha, \beta) - \sum_{n=2}^{\infty} \left[\frac{1 + (-1)^n}{n^2 - 1} \right] B_n(\alpha, \beta). \tag{3.11}$$

For practical computations, this infinite series is truncated at a certain order m_1.

The optimal control problem is now reduced to a parameter optimization problem which is stated as follows:

> Find α and β that minimize $J(\alpha, \beta)$ subject to the constraint relations (3.7) and (3.8) which may be written as

$$F_\nu(\alpha, \beta) = 0, \qquad (\nu = 1, 2, \ldots, m + 5). \tag{3.12}$$

By applying the method of constrained extremum which consists of adjoining the constraints to the performance index by a set of undetermined Lagrange multipliers λ_ν, $(\nu = 1, 2, \ldots, m + 5)$, and expressing the necessary conditions for a stationary value of $J(\alpha, \beta)$, one obtains the following determining equations for the unknowns α, β, and λ:

$$\frac{\partial J}{\partial a_\mu} + \lambda_\nu \frac{\partial F}{\partial a_\mu} = 0, \qquad \frac{\partial J}{\partial b_\mu} + \lambda_\nu \frac{\partial F_\nu}{\partial b_\mu} = 0, \qquad F_\nu(\alpha, \beta) = 0 \tag{3.13}$$

$(\mu = 0, 1, \ldots, m;\ \nu = 1, 2, \ldots, m + 5)$.

Sufficient conditions for a local minimum are the stationarity conditions (first and second groups of equations in (3.13)) and the convexity condition expressing the positive definiteness of a certain matrix [16].

In the case of the controlled oscillator as considered here, the function g is proportional to u^2, and therefore J depends only upon β. Since the system dynamics is described by a linear function in x and u, and the performance index is characterized by a quadratic function in u, the determining equations (3.13) are linear in all unknowns α, β, and λ, and they are readily solved by any classical method in numerical analysis such as Gauss elimination or LU triangular decomposition with pivoting.

The computation of the Chebyshev coefficients B_n given in Eq. (3.10) is carried out as follows. Putting $t = \cos\theta$ and using the property $T_n(\cos\theta) = \cos n\theta$, the Chebyshev coefficients B_n (where $n = 0, 1,..., m_1$) may be computed by the following approximation formulae [8]

$$B_n = \frac{2}{N} \sum_{i=1}^{N} g[x_m(\cos\theta_i), u_m(\cos\theta_i), \cos\theta_i] \cos n\theta_i \qquad (n = 0, 1,..., m_1)$$

$$N > m_1, \quad \theta_i = [(2i-1)/2N]\,\pi. \qquad (3.14)$$

It has been pointed out that the Chebyshev approximation of relatively low order $m = 4$ may be readily obtained in a complete analytical manner. In that case, there are nine constraint equations between the ten unknown Chebyshev coefficients. The results are reported in the first column of Table I with the following choice of the numerical values of the parameters:

$$\omega = 1, \qquad T = 2, \qquad x_{-1} = 0.5, \qquad \dot{x}_{-1} = -0.5. \qquad (3.15)$$

The same results were obtained numerically using a CDC CYBER 170/750 computer by taking $m_1 = 6$ and $N = 12$. Higher order Chebyshev approximations have then been computed and the results for the Chebyshev approximations of order $m = 7$ and $m = 10$ have been incorporated in Table I. The corresponding values of the parameters m_1 and N which have been chosen are listed, and the computational time (TIME) is indicated.

The Chebyshev coefficients for the exact state function $x(t)$ and the exact control strategy $u(t)$ given in Eq. (2.4) have been computed from (3.14) and are listed between brackets in the last column in Table I. One of the most important advantages of the use of the Chebyshev series representation is that the Chebyshev coefficients rapidly decrease. Generally other expansions need more coefficients if the same precision is required. As a consequence, the computational time increases considerably. By proceeding to the higher order approximations, the results obtained by the Chebyshev technique gradually tend to the results for the exact solution representation. The Chebyshev approximation of order $m = 10$ is already a very accurate approximation of the exact solution. The largest deviation in the coefficients amounts to 1.2×10^{-8} and there is an agreement of 14 decimal places for J.

TABLE I

The Chebyshev Approximations for the Controlled Linear Oscillator

	Chebyshev solution $m = 4$ $m_1 = 6, N = 12$	Chebyshev solution $m = 7$ $m_1 = 10, N = 20$	Chebyshev solution $m = 10$ $m_1 = 15, N = 30$
a_0	0.351120	0.35111171	0.3511116502 (02)
a_1	−0.250000	−0.24971581	−0.2497158104 (04)
a_2	0.078420	0.07851606	0.0785159994 (94)
a_3	0.0	−0.00042883	−0.0004288315 (15)
a_4	−0.003980	−0.00412624	−0.0041265537 (37)
a_5		0.00014634	0.0001463517 (17)
a_6		0.00005432	0.0000550660 (60)
a_7		−0.00000170	−0.0000017187 (87)
a_8			−0.0000003380 (80)
a_9			0.0000000089 (89)
a_{10}			0.0000000012 (12)
b_0	0.723762	0.72689438	0.7268625864 (59)
b_1	−0.250000	−0.24301769	−0.2430166656 (99)
b_2	−0.112618	−0.10911365	−0.1091470156 (61)
b_3	0.0	0.01080270	0.0108038104 (059)
b_4	−0.003980	0.00239238	0.0023525641 (36)
b_5		−0.00013927	−0.0001379230 (80)
b_6		0.00005432	−0.0000198888 (94)
b_7		−0.00000170	0.0000008358 (297)
b_8			0.0000000903 (895)
b_9			(−) 0.0000000089 (29)
b_{10}			(−) 0.0000000012 (02)
J	0.184917	0.18485854	0.1848585424 (24)
TIME	0.32 sec	0.74 sec	1.8 sec

TABLE II

Error versus Parameter Variations

	Maximum error on a_n	Maximum error on b_n	Error on J
Standard case	5.3×10^{-11}	1.2×10^{-8}	2.7×10^{-15}
$\omega = 2$	7.3×10^{-8}	1.6×10^{-5}	9.7×10^{-11}
$T = 3$	3.5×10^{-9}	3.4×10^{-7}	6.6×10^{-14}
$x_{-1} = 1, \dot{x}_{-1} = -1$	1.1×10^{-10}	2.4×10^{-8}	1.1×10^{-14}

Table II represents, for various values of the parameters ω, T, x_{-1}, and $\dot{x}_{-1}$, the maximum error on the Chebyshev coefficients of order $m = 10$ and on the performance index in comparison with the results obtained from the exact solution representation. By increasing the value of some of these parameters, holding the other parameters at constant values, it has been found that the Chebyshev approximations are less accurate.

4. The Controlled Duffing Oscillator

Let us now investigate the optimal control of the Duffing oscillator described by the nonlinear differential equation

$$\ddot{x} + \omega^2 x + \varepsilon x^3 = u \qquad (\dot{} = d/d\tau), \tag{4.1}$$

subject to the same boundary conditions as before and taking the same performance index expression. Of course, the exact solution in this case is not known.

By applying the Chebyshev technique introduced in Section 3, the following modifications have to be taken into account. The approached system dynamics becomes

$$\ddot{x}_m(t) = f_m[x_m(t), u_m(t)] \qquad (\dot{} = d/dt), \tag{4.2}$$

where

$$f(x, u) = \tfrac{1}{4}T^2(-\omega^2 x - \varepsilon x^3 + u), \tag{4.3}$$

and in which $f_m[x_m(t), u_m(t)]$ represents the Chebyshev series of $f[x_m(t), u_m(t)]$ truncated after the term of order m. The approached system dynamics, boundary conditions, and performance index take the same expressions as Eqs. (3.7), (3.8), and (3.11), at least formally. The coefficients A_n in Eq. (3.7), however, are now nonlinear functions in α and β, and hence the determining equations (3.13) for the unknowns α, β, and λ are also nonlinear. Starting with some initial values $\bar{\alpha}$, $\bar{\beta}$, and $\bar{\lambda}$, the generalized Newton–Raphson iterative method will be applied to solve these equations.

The starting values $\bar{\alpha}$ and $\bar{\beta}$ have been taken from the analytical treatment with $m = 4$ for the controlled linear oscillator ($\varepsilon = 0$) which has been discussed previously. Starting values for λ_ν ($\nu = 1, 2, \ldots, m + 5$) were obtained by selecting $m + 5$ equations from the first two sets of equations in (3.13). In this way a linear system with respect to λ_ν is obtained once initial values for α and β are given.

Table III lists the results for the Chebyshev approximations of order $m = 4$, 7, and 10 for the same numerical values of the parameters ω, T, x_{-1}, and $\dot{x}_{-1}$ as given in (3.15) and where, in addition, the coefficient ε of the nonlinearity has been taken as $\varepsilon = 0.15$. The precision on α, β, and λ imposed in order to stop Newton's iterative method is indicated by PREC, and ITER represents the number of iterations required

TABLE III

The Chebyshev Approximations for the Controlled Duffing Oscillator

	Chebyshev solution $m = 4$ $m_1 = 6, N = 12$	Chebyshev solution $m = 7$ $m_1 = 10, N = 20$	Chebyshev solution $m = 10$ $m_1 = 15, N = 30$
a_0	0.350240	0.35023304	0.3502330049
a_1	−0.250000	−0.24963104	−0.2496310664
a_2	0.079007	0.07909029	0.0790902898
a_3	0.0	−0.00055515	−0.0005551413
a_4	−0.004127	−0.00425420	−0.0042543424
a_5		0.00018731	0.0001872655
a_6		0.00004739	0.0000476871
a_7		−0.00000113	−0.0000009808
a_8			−0.0000001464
a_9			−0.0000000771
a_{10}			0.0000000096
b_0	0.726323	0.72904647	0.7290361297
b_1	−0.257098	−0.24794900	−0.2479592161
b_2	−0.114407	−0.11134470	−0.1113555543
b_3	−0.002218	0.01188669	0.0118753763
b_4	−0.003446	0.00211863	0.0021059922
b_5		−0.00007899	−0.0000924297
b_6		0.00000900	−0.0000173724
b_7		0.00001834	−0.0000037004
b_8			0.0000007451
b_9			−0.0000006837
b_{10}			0.0000002377
J	0.187531	0.18744484	0.1874448561
PREC	10^{-6}	10^{-8}	10^{-10}
ITER	2	3	3
TIME	0.79 sec	1.9 sec	4.7 sec

to attain such precision. Similar conclusions as formulated for the linear case hold here. In addition, the effect of the parameter ε characterizing the nonlinearity has been studied. When ε is increased, one has to take a larger value of the order m of the Chebyshev approximations in order to obtain the same precision.

References

1. A. P. Sage and C. C. White III, "Optimum Systems Control," Prentice–Hall, Englewood Cliffs, N. J., 1977.

2. D. A. WISMER AND R. CHATTERGY, "Introduction to Nonlinear Optimization," North-Holland, Amsterdam, 1978.
3. L. C. W. DIXON AND G. P. SZEGÖ, "Numerical Optimisation of Dynamic Systems," North-Holland, New York, 1980.
4. R. BELLMAN, "Dynamic Programming," Princeton Univ. Press, Princeton, N. J., 1957.
5. L. S. PONTRYAGIN, V. BOLTYANSKII, R. GAMKRELIDZE AND E. MISCHENKO, "The Mathematical Theory of Optimal Processes," Interscience, New York, 1962.
6. J. C. GILLE, P. DECAULNE AND M. PELEGRIN, "Systèmes Asservis Non Linéaires," Bordas, Paris, 1975.
7. A. H. NAYFEH AND D. T. MOOK, "Nonlinear Oscillations," Wiley, New York, 1979.
8. L. FOX AND I. B. PARKER, "Chebyshev Polynomials in Numerical Analysis," Oxford Univ. Press, Oxford, 1972.
9. M. URABE, *in* "Lecture Notes in Mathematics" (Dold, Ed.), No. 109, pp. 40–86, Springer-Verlag, Berlin/New York, 1969.
10. R. VAN DOOREN, *Numer. Math.* **20** (1973), 300.
11. R. VAN DOOREN, *Int. J. Non-Linear Mech.* **8** (1973), 575.
12. R. VAN DOOREN, *J. Comput. Phys.* **16** (1974), 186.
13. R. VAN DOOREN, *J. Sound Vib.* **41** (1975), 133.
14. R. VAN DOOREN, *Acad. R. Belg. Bull. Cl. Sci., 5e série,* **64** (1978), 360.
15. G. DAHLQUIST AND A. BJORCK, "Numerical Methods," Printice–Hall, Englewood Cliffs, N. J., 1974.
16. A. E. BRYSON AND Y. C. HO, "Applied Optimal Control," Ginn (Blaisdell), Boston, 1969.

RECEIVED: November 24, 1981

R. VAN DOOREN AND J. VLASSENBROECK

Department of Analytical Mechanics
Vrije Universiteit Brussel
Brussels, Belgium